Original Article

IJEEE

International Journal of Electrical Engineering Education
2015, Vol. 52(2) 131–141
© The Author(s) 2015
Reprints and permissions:
sagepub.co.uk/journalsPermissions.nav
DOI: 10.1177/0020720915571799
ije.sagepub.com
SAGE

Analytic solutions of electromagnetic fields in inhomogeneous media

Slobodan V Savić, Aleksandra Krneta, Marija Stevanović, Dragan I Olćan, Miodrag Tasić, Milan M Ilić, Dejan Tošić, Branko Kolundžija and Antonije R Djordjević

Abstract

We present guidelines for teaching students how to analytically solve problems that involve inhomogeneous media in electrostatic fields, stationary current fields, and stationary magnetic fields. At the introductory level, the focus is on recognizing classes of problems that can be solved in closed form and applying simple rules, based on comparison with solutions in homogeneous media. At the intermediate level, the focus is on strict proofs based on vector calculus.

Keywords

Fundamentals of electrical engineering, electromagnetics, analytic solutions, problems with inhomogeneous media

Introduction

There are many efficient numerical techniques for analysis of complex electromagnetic (EM) problems. In some curricula they are introduced to students as early as at the undergraduate level, as in Haldar,[1] and most of them are available as commercial software. In addition, modern EM courses are often application driven and/or based on some form of computer aided education tools, as in Beker et al.[2] However, the numerical methods usually do not provide sufficient insight into the underlying physics needed for mentally grasping the problem. Further, they are usually inappropriate for the introduction of new ideas and concepts, particularly to undergraduate students. Excessive reliance on modern software

School of Electrical Engineering, University of Belgrade, Serbia

Corresponding author:
Slobodan V Savić, School of Electrical Engineering, University of Belgrade, Serbia.
Email: ssavic@etf.rs

tools and the black-box approach practically create deficiency in understanding various physical phenomena and input–output relations of systems under consideration, as studied in Collin[3] and Siakavellas.[4]

Present-day students of electrical engineering are facing two major challenges. The first is to successfully complete five-year BSc/MSc studies and adopt electrical engineering fundamentals. The second challenge is getting ready to cope with real-life structures that impose understanding of complex problems (such as analysis and design of EM systems) and to be prepared for the future technologies, about which we can only have an educated guess. Hence, we believe that it is necessary for the students in electrical engineering to acquire solid knowledge of the EM-field theory as early as at the freshmen level of undergraduate studies. Good fundamentals are essential prerequisites for later education in electrical machines, electronics, microwaves, antennas and radio-wave propagation, solid-state physics, optics, nanotechnologies, etc. However, teaching the EM-field theory is not easy, because the students are exposed to new concepts and the mathematical skills are demanding.

At the undergraduate level, it is appropriate to consider EM-field problems that have analytical solutions. To get started with understanding of these problems, students are traditionally exposed to selected simple examples of EM structures, which can be solved by hand. In other words, the most natural way, in our opinion, to qualify and quantify these problems is to solve some classes of problems analytically, and obtain closed-form solutions, which, in turn, provide insight into studied phenomena. Examples are the electrostatic field of a charged metallic sphere in a vacuum and the magnetic field of an infinitely long, straight current filament.

Our experience is that the students can relatively easily apprehend analysis of EM structures with homogeneous media. However, difficulties arise with problems that involve inhomogeneous media, such as the electrostatic field in a coaxial cable with a layered dielectric. Such problems are important in engineering practice; thus the students should be able to analyze at least some basic cases.

At the introductory level, where the students are not yet familiar with the vector calculus, it is practically impossible to derive rigorous analytical solutions for EM-field problems in inhomogeneous media. However, we have established some simple rules on how to solve such problems, based on comparisons with problems with homogeneous media. At the intermediate level, in the courses in EM fields, the students (who elect appropriate courses) are taught rigorous proofs of these simple rules.

The goal of this paper is to present teaching guidelines, at both the introductory and intermediate levels, for the analytical solutions of EM-field problems in inhomogeneous media. For better understanding of the teaching challenges, in "Undergraduate curriculum" section we outline the undergraduate curriculum at our school. In "Electrostatics" section, the analysis of electrostatic problems is presented in detail. This approach is directly translated to the analysis of stationary current fields. In "Stationary magnetic field" section, the analysis of stationary magnetic fields is discussed. Finally, in "Conclusion" section, the success of the teaching method is assessed.

Undergraduate curriculum

The undergraduate curriculum at the School of Electrical Engineering of the University of Belgrade is organized with the idea to give solid basic theoretical knowledge. In each semester of the first academic year, all our students (except those who elect pure software engineering) have two obligatory courses entitled Fundamentals of Electrical Engineering 1 and 2, respectively. Each course has 3 h of lectures and 3 h of recitations per week (7 ECTS each). These courses cover EM fields and circuits: electrostatic fields, stationary current fields, and direct-current (DC) circuits in the first semester, and magnetic fields (stationary and quasistationary) and alternating-current (AC) circuits and transients in the second semester with details in Djordjević[5] and Božilović et al.[6] The presentation of EM fields concludes with Maxwell's equations in the integral form. The goal of these two courses, in general, is to introduce the basic concepts, equations, and analysis methods for circuits and EM fields.

There are several teaching challenges at this level. First, students come from various high schools and their skills in mathematics and physic are highly different. A particular problem is the level of knowledge in calculus. Second, the number of students who attend these courses is relatively large (more than 500), so it is not easy to devote sufficient attention to all of them. Third, in a relatively short allocated time, the courses must provide solid theoretical background and also provide the students with skills to independently solve various engineering problems.

In the second or third academic year, most students take a course in EM fields (one semester). In this course, an in-depth analysis of EM fields is presented, using vector calculus, as studied in Djordjević[7] and Notaroš.[8] Engineering concepts and ideas are introduced as required for the understanding of radio systems, microwave and optical communication systems, fast digital circuits, and microelectronics. Maxwell's equations in the integral and differential form in the time domain and in the frequency domain are considered, followed by the analysis of plane waves, guided waves, transients on transmission lines, antennas, and an introduction to EM compatibility.

Electrostatics

At the beginning of Fundamentals of Electrical Engineering 1, electrostatic fields are considered. The basic integral expressions for the electrostatic field vector ($\mathbf{E}_0$) and electrostatic potential (V_0) in a vacuum are considered, including Gauss's law.

Among other issues, Gauss's law is used to obtain analytical solutions for the vector $\mathbf{E}_0$ in a vacuum in problems with a high degree of symmetry. There are three special cases for which such closed-form solutions are considered. In each case, due to symmetry, it is possible to a priori determine the direction of the vector $\mathbf{E}_0$ and the coordinate on which its intensity depends.

In the first case (spherical geometry), the charge distribution depends only on the distance from one point. If we place the origin of a spherical coordinate system at that point, due to symmetry, the vector $\mathbf{E}_0$ can have only the radial (r) component, which depends on the r-coordinate. Examples are a point charge and a sphere uniformly charged over its surface.

In the second case (cylindrical geometry), the charge distribution depends only on the distance from one line. If that line coincides with the z-axis of a cylindrical coordinate system, the vector $\mathbf{E}_0$ can have only the radial (r) component, which depends on the r-coordinate. An example is a uniformly charged, very long, straight filament.

Finally, in the third case (planar geometry), the charge distribution depends only on the distance from one plane. If that plane coincides with the Oyz plane of a Cartesian coordinate system, the vector $\mathbf{E}_0$ can have only the x-component, which depends on the x-coordinate. An example is a uniformly charged plane.

In all three cases, an analytical solution can be found by choosing an appropriate closed surface and applying Gauss's law to it.

The students usually easily grasp solutions to these problems.

Thereafter, dielectrics are introduced and the generalized Gauss's law (Gauss's law for vector **D**) is derived. Solved problems include structures with linear inhomogeneous dielectrics. Some examples are shown in Figure 1. In all cases, we have one or more electrodes. In some cases, the dielectric is piecewise-homogeneous, whereas in other cases, the permittivity continuously varies in space.

This is the point where many students struggle. To help them, we have designed a set of guidelines to understand (i) which problems can have an analytical solution and (ii) how to cope with such problems. These guidelines are given without proofs; the proofs are postponed for the EM course.

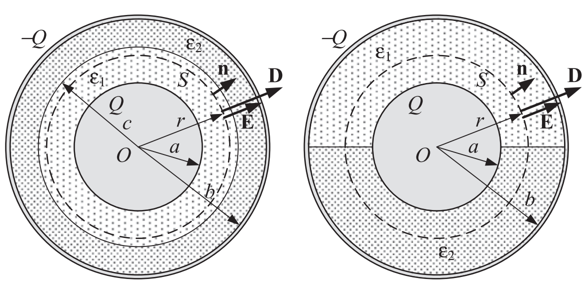

Figure 1. Cross-section of a spherical capacitor with inhomogeneous dielectric as in Djordjević[5] and Božilović et al.[6] (a) Permittivity varies only along the lines of the vector $\mathbf{E}_0$ and the structure of the vector **D** is the same as in the reference problem. (b) Permittivity varies only over the equipotential surfaces and the structure of the vector **E** is the same as in the reference problem.

(i) We tell the students to consider the given structure in a vacuum, i.e. to imagine that the dielectric is removed. The geometry of electrodes in a vacuum (the reference problem) should conform to one of the three geometries specified above (spherical, cylindrical, or planar), for which the students already know how to determine the field distribution. For this reference problem, the students should visualize the electric-field vector $\mathbf{E}_0$ and the corresponding electric flux density vector $\mathbf{D}_0 = \varepsilon_0 \mathbf{E}_0$. With respect to the reference problem, the dielectric properties can vary only in one of the two special ways:

- The dielectric permittivity varies only along the lines of the vector $\mathbf{E}_0$, and it is constant over any equipotential surface (which is perpendicular to the lines of the vector $\mathbf{E}_0$). In this case, the structure (field) of the vector $\mathbf{D}$ is the same as in the reference problem. An example is the problem illustrated in Figure 1(a).
- The dielectric permittivity varies only over equipotential surfaces, and it is constant along any line of the vector $\mathbf{E}_0$. In this case, the structure of the vector $\mathbf{E}$ is the same as in the reference problem. An example is the problem illustrated in Figure 1(b).

(ii) Based on the above considerations, we know the structure of the vector $\mathbf{D}$, viz. $\mathbf{E}$, which is of paramount importance for the remaining solution procedure. Knowing the field structure, the next hint is to use the generalized Gauss's law and carry on with subsequent derivations.

As a matter of explanation, we tell the students, without a proof, that the assumed vectors $\mathbf{D}$ and $\mathbf{E}$, in both cases, automatically satisfy the basic integral equations for the electrostatic fields and the corresponding boundary conditions, so that the solution must be unique.

As two typical examples for this procedure, we determine the capacitance of spherical capacitors with inhomogeneous dielectric shown in Figure 1(a) and (b). The reference problem belongs to the spherical geometry class of problems. By applying Gauss's law, we easily determine the vectors $\mathbf{E}_0 = \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{i}_r$ and $\mathbf{D}_0 = \frac{Q}{4\pi r^2}\mathbf{i}_r$ in a vacuum (between the electrodes).

For the first example shown in Figure 1(a), the dielectric permittivity varies only along the lines of the vector $\mathbf{E}_0$. The structure of the vector $\mathbf{D}$ is the same as in the reference problem, i.e. $\mathbf{D} = \mathbf{D}_0 = \frac{Q}{4\pi r^2}\mathbf{i}_r$. Hence, $\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_1 r^2}\mathbf{i}_r,\ a < r < c$ and $\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_2 r^2}\mathbf{i}_r,\ c < r < b$. The capacitor voltage is $U = \int_a^b \mathbf{E}\cdot d\mathbf{l} = \frac{Q}{4\pi}\left(\frac{1}{\varepsilon_1}\frac{c-a}{ac} + \frac{1}{\varepsilon_2}\frac{b-c}{cb}\right)$ and the capacitance is $C = \frac{Q}{U} = \frac{4\pi}{\frac{1}{\varepsilon_1}\frac{c-a}{ac} + \frac{1}{\varepsilon_2}\frac{b-c}{cb}}$.

For the second example, shown in Figure 1(b), the dielectric permittivity varies only over the equipotential surfaces. The structure of the vector $\mathbf{E}$ is the same as in the reference problem, i.e. it is radial and its intensity depends only on r. Now, $\mathbf{D} = \varepsilon_1 \mathbf{E}$ in the first dielectric and $\mathbf{D} = \varepsilon_2 \mathbf{E}$ in the second dielectric. By applying Gauss's law, we determine the electric-field vector between the capacitor electrodes

$\mathbf{E}(\mathbf{r}) = \frac{Q}{2\pi r^2(\varepsilon_1+\varepsilon_2)}\mathbf{i}_r$. The capacitor voltage is $U = \int_a^b \mathbf{E}\cdot d\mathbf{l} = \frac{Q}{2\pi(\varepsilon_1+\varepsilon_2)}\frac{b-a}{ab}$ and the capacitance is $C = \frac{Q}{U} = \frac{2\pi(\varepsilon_1+\varepsilon_2)ab}{b-a}$.

The proof, given in the EM course, runs as follows. We start from the reference electrostatic problem (in a vacuum), which consists of N ($N \geq 1$) metallic bodies whose charges are $Q_{0,1}, Q_{0,2}, \ldots, Q_{0,N}$ and potentials are $V_{0,1}, V_{0,2}, \ldots, V_{0,N}$, respectively. We assume that we know the solution for this problem: the electric-field vector $\mathbf{E}_0(\mathbf{r})$ and the electric flux density vector $\mathbf{D}_0(\mathbf{r}) = \varepsilon_0\mathbf{E}_0(\mathbf{r})$. In the space outside the metallic bodies, these vectors satisfy the differential equations for the electrostatic field, curl $\mathbf{E}_0 = 0$ and div $\mathbf{D}_0 = 0$, along with the boundary conditions $\mathbf{n}_n \times \mathbf{E}_{0,n} = 0$ and $\mathbf{n}_n \cdot \mathbf{D}_{0,n} = \rho_{s0,n}$ ($n = 1, \ldots, N$) at the surfaces of the metallic bodies, where $\mathbf{n}_n$ is the corresponding outward normal, and $\rho_{s0,n}$ is the surface-charge density on the nth metallic body.

Further, we consider an electrostatic problem consisting of metallic bodies with the same shape as in the reference problem, but where the free-space is replaced by a linear, inhomogeneous dielectric, whose relative permittivity is $\varepsilon_r(\mathbf{r})$. In this problem, the electric-field vector is $\mathbf{E}(\mathbf{r})$ and the flux density vector is $\mathbf{D}(\mathbf{r}) = \varepsilon_r(\mathbf{r})\varepsilon_0\mathbf{E}(\mathbf{r})$. The main concern is to establish which spatial variation of $\varepsilon_r(\mathbf{r})$ yields (a) $\mathbf{D}(\mathbf{r}) = \mathbf{D}_0(\mathbf{r})$ and (b) $\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r})$.

In both cases, the vectors $\mathbf{E}(\mathbf{r})$ and $\mathbf{D}(\mathbf{r})$ must satisfy the differential equations for the electrostatic field, curl $\mathbf{E} = 0$ and div $\mathbf{D} = 0$ within the dielectric, as well as the boundary conditions $\mathbf{n}_n \times \mathbf{E}_n = 0$ and $\mathbf{n}_n \cdot \mathbf{D}_n = \rho_{s,n}$ at surfaces of the electrodes. Due to the uniqueness theorem for electrostatic fields in inhomogeneous media, the resulting solution must be unique, as studied in Djordjević.[7]

(a) In the case when $\mathbf{D}(\mathbf{r}) = \mathbf{D}_0(\mathbf{r})$, the equation div $\mathbf{D} = 0$ is automatically satisfied, because div $\mathbf{D}_0 = 0$ in the reference problem. The surface-charge density of the metallic bodies is the same as in the reference problem, since $\mathbf{n}_n \cdot \mathbf{D}_n = \mathbf{n}_n \cdot \mathbf{D}_{0,n} = \rho_{s,n} = \rho_{s0,n}$. In this case, $\mathbf{E} = \frac{\mathbf{E}_0}{\varepsilon_r} \neq \mathbf{E}_0$. From curl $\mathbf{E} = 0$, it follows that $\mathrm{curl}\left(\frac{\mathbf{E}_0}{\varepsilon_r}\right) = \left(\mathrm{grad}\left(\frac{1}{\varepsilon_r}\right)\right) \times \mathbf{E}_0 + \frac{1}{\varepsilon_r}\mathrm{curl}\,\mathbf{E}_0 = \left(\mathrm{grad}\left(\frac{1}{\varepsilon_r}\right)\right) \times \mathbf{E}_0 = 0$. This will be satisfied everywhere if and only if $\left(\mathrm{grad}\left(\frac{1}{\varepsilon_r}\right)\right) || \mathbf{E}_0$. Hence, the permittivity can vary only in directions collinear with $\mathbf{E}_0$ and $\mathbf{D}_0$ (i.e. along the lines of the vectors $\mathbf{E}_0$ and $\mathbf{D}_0$). (A homogeneous dielectric, for which $\varepsilon_r = \mathrm{const}$, is a special case.) The surface-charge density of metallic bodies is the same as in the reference electrostatic analysis problem ($Q_n = Q_{0,n}$) since $\mathbf{D}(\mathbf{r}) = \mathbf{D}_0(\mathbf{r})$, but the potentials are different than in the reference problem ($V_n \neq V_{0,n}$) since $\mathbf{E} \neq \mathbf{E}_0$.

(b) In the case when $\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r})$, equations curl $\mathbf{E} = 0$ and $\mathbf{n}_n \times \mathbf{E}_n = 0$ are automatically satisfied, because curl $\mathbf{E}_0 = 0$ and $\mathbf{n}_n \times \mathbf{E}_{0,n} = 0$ in reference problem. Further, $\mathbf{D} = \varepsilon_r(\mathbf{r})\mathbf{D}_0 \neq \mathbf{D}_0$. From the differential equation div $\mathbf{D} = 0$, it follows that $\mathrm{div}(\varepsilon_r\mathbf{D}_0) = (\mathrm{grad}\,\varepsilon_r) \cdot \mathbf{D}_0 + \varepsilon_r \mathrm{div}\,\mathbf{D}_0 = (\mathrm{grad}\,\varepsilon_r) \cdot \mathbf{D}_0 = 0$. This will be satisfied at each point if and only if $\mathbf{D}_0 \perp \mathrm{grad}\,\varepsilon_r$. Hence, the permittivity can vary only in directions perpendicular to $\mathbf{D}_0$ and $\mathbf{E}_0$. In other words, the permittivity can vary only over equipotential surfaces. (A homogeneous dielectric, for which $\varepsilon_r = \mathrm{const}$, is a special case.) The potentials of the metallic bodies are the same as in the reference problem ($V_n = V_{0,n}$) since $\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r})$. However, the electrode charges are different than in the reference problem ($Q_n \neq Q_{0,n}$) since $\mathbf{D} \neq \mathbf{D}_0$.

Note that similar reasoning applies to the analysis of problems related to the distribution of stationary currents in inhomogeneous media.

Stationary magnetic field

We consider here the evaluation of the distribution of magnetic field due to stationary electric currents. The medium is assumed to be linear, but inhomogeneous. As for the electrostatic fields in the previous section, the key question is about possible variations of the relative permittivity of the medium (μ_r) that enable analytical solutions.

Following the reasoning from the previous section, we consider first the magnetic field in a vacuum. There are three special geometries for which the magnetic flux density $\mathbf{B}_0$ can be analytically determined starting only from Ampere's law.

In the first (cylindrical) geometry, the distribution of the electric current is rotationally symmetric and has only the component parallel to the axis of symmetry. If we place a cylindrical coordinate system so that the z-axis coincides with the axis of symmetry, vector $\mathbf{B}_0$ can have only the ϕ-component, which depends on the r-coordinate. An example is an infinite, straight current filament.

In the second (toroidal) geometry, the electric currents flow around generatrices of a toroidal surface. If the axis of the toroid coincides with the z-axis of a cylindrical coordinate system, vector $\mathbf{B}_0$ can have only the ϕ-component, which depends on the r-coordinate inside the toroid and is zero elsewhere. An example is a winding on a toroidal transformer.

In the third (planar) geometry, the distribution of the electric current depends only on the distance from a plane. If this plane coincides with the Oyz plane of a Cartesian coordinate system, and the electric current–density vector $\mathbf{J}_0$ has only the z-component whose intensity depends on the x-coordinate, i.e. $\mathbf{J}_0 = J_{0,z}(x)\,\mathbf{i}_z$, the magnetic flux density vector can have only the y-component, which depends on x, i.e. $\mathbf{B}_0 = B_{0,y}(x)\,\mathbf{i}_y$. An example is an infinite, flat sheet of uniform surface electric currents.

For each of these three geometries, by applying Ampere's law on an appropriately chosen contour, we can determine the distribution of the magnetic field.

Thereafter, magnetic materials are introduced and the generalized Ampere's law (Ampere's law for vector $\mathbf{H}$) is derived. Solved problems include structures with linear inhomogeneous magnetic materials. Some examples are shown in Figure 2. In some cases, the magnetic material is piecewise-homogeneous, whereas in other cases, the permeability continuously varies in space.

In order to help students solve these field problems, we give guidelines to understand (i) which problems can have an analytical solution and (ii) how to cope with such problems. These guidelines are given without a proof, and again the proof is

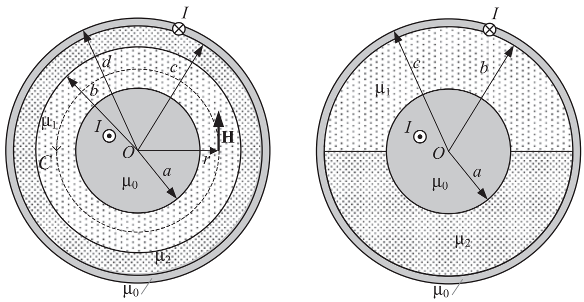

Figure 2. Cross-section of a coaxial cable with inhomogeneous magnetic material, as discussed in Djordjević[5] and Božilović et al.[6] (a) Permeability varies only in the direction perpendicular to $\mathbf{B}_0$ and $\mathbf{H}_0$ and the structure of the vector $\mathbf{H}$ is the same as in the reference problem. (b) Permeability varies in a complex manner: it varies in the direction perpendicular to $\mathbf{B}_0$ and $\mathbf{H}_0$ on the boundary between the conductor and the magnetic material, and in the direction collinear with $\mathbf{B}_0$ and $\mathbf{H}_0$ at the boundary between two magnetic materials. There is no analytical solution for this type of inhomogeneity.

postponed for the EM course. Since students have some experience with electrostatics, usually they now easily adopt these guidelines.

(i) We start from the given geometry of conductors in a vacuum (the reference problem). This geometry should conform to one of the three geometries specified above (cylindrical, toroidal, or planar), for which students already know how to determine the field distribution. With respect to the reference problem, the magnetic properties can vary only in one of the two special ways:

- The permeability varies only in directions perpendicular to $\mathbf{B}_0$ and $\mathbf{H}_0$. In this case, the structure of the vector $\mathbf{H}$ is the same as in the reference problem. An example is shown in Figure 2(a).
- The permeability varies only in directions collinear with $\mathbf{B}_0$ and $\mathbf{H}_0$. In this case, the structure of the vector $\mathbf{B}$ is the same as in the reference problem. However, the current distribution must be different than the current distribution in the reference problem. This can be quite difficult to understand and visualize for first year undergraduate students, so that we postpone this type of problems to senior years and EMs course.

(ii) For the given problem with an inhomogeneous magnetic material, we now know the structure of vector $\mathbf{B}$, viz. $\mathbf{H}$. Knowing the field structure, the next step is to use the generalized Ampere's law and carry on with subsequent derivations.

The assumed vectors $\mathbf{B}$ and $\mathbf{H}$ automatically satisfy the basic integral equations for the stationary magnetic field and the corresponding boundary conditions, so that the solution must be unique, as studied in Djordjević.[7] If the problem does not belong to one of the two cases, the analytical solution cannot be determined by applying only Ampere's law. An example of a problem with no analytical solution is shown in Figure 2(b), where the current I is uniformly distributed over the conductor cross-sections.

As examples, we consider coaxial cables with inhomogeneous magnetic materials shown in Figure 2. The reference problem belongs to the cylindrical geometry, so that vector $\mathbf{H}_0$ has only the ϕ-component. By applying Ampere's law, we can easily determine magnetic field distribution inside coaxial cable: $H_0(\mathbf{r}) = \frac{Ir}{2\pi a^2},\ r < a$, $H_0(\mathbf{r}) = \frac{I}{2\pi r},\ a \le r \le c,\ H_0(\mathbf{r}) = \frac{I(d^2 - r^2)}{2\pi r(d^2 - c^2)},\ c < r < d$, and $H_0(\mathbf{r}) = 0,\ r \ge d$. Since we consider the conductor to be non-magnetic, we have $B_0(\mathbf{r}) = \mu_0 H_0(\mathbf{r})$ everywhere.

For the first example, shown in Figure 2(a), the permeability varies only in the direction perpendicular to $\mathbf{B}_0$ and $\mathbf{H}_0$. The structure of vector $\mathbf{H}$ is the same as in the reference problem. The vector $\mathbf{B}$ is obtained from the constitutive relation; it has only the ϕ-component, $B(\mathbf{r}) = \frac{\mu_0 Ir}{2\pi a^2},\ r < a,\ B(\mathbf{r}) = \frac{\mu_1 I}{2\pi r},\ a < r < b,\ B(\mathbf{r}) = \frac{\mu_2 I}{2\pi r}$, $b < r < c,\ B(\mathbf{r}) = \frac{\mu_0 I(d^2 - r^2)}{2\pi r(d^2 - c^2)},\ c < r < d$, and $B(\mathbf{r}) = 0,\ r \ge d$.

For the second example, shown in Figure 2(b), the permeability varies in a complex manner. It varies in the direction perpendicular to $\mathbf{B}_0$ and $\mathbf{H}_0$ on the boundaries between each conductor and the magnetic materials, as well as in the direction collinear with $\mathbf{B}_0$ and $\mathbf{H}_0$ at the boundary between two magnetic materials. Since this problem does not belong to either of the two above-mentioned inhomogeneous types, analytical solution cannot be determined by the proposed procedure.

Next, we briefly present the mathematical proof for these guidelines.

Knowing the solution for a problem in a vacuum, we are interested in finding the corresponding problem with a linear, inhomogeneous, magnetic material, which has a closed-form solution for the distribution of the magnetic field.

We start from the reference problem in which the electric currents $\mathbf{J}_0(\mathbf{r})$ are situated in a vacuum, and vectors $\mathbf{B}_0(\mathbf{r})$ and $\mathbf{H}_0(\mathbf{r})$ are known. This magnetic field must satisfy the differential equations $\operatorname{curl} \mathbf{H}_0 = \mathbf{J}_0$ and $\operatorname{div} \mathbf{B}_0 = 0$, as well as the constitutive relation $\mathbf{B}_0 = \mu_0 \mathbf{H}_0$, where μ_0 is the free-space permeability.

We now consider the corresponding magnetic problem in which the free-space has been replaced by a linear and inhomogeneous magnetic material whose relative permeability is $\mu_r(\mathbf{r})$. The magnetic field is determined by $\mathbf{H}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r}) = \mu_r(\mathbf{r})\mu_0 \mathbf{H}(\mathbf{r})$. We want to know which spatial variations of $\mu_r(\mathbf{r})$ yield (a) $\mathbf{H}(\mathbf{r}) = \mathbf{H}_0(\mathbf{r})$ and (b) $\mathbf{B}(\mathbf{r}) = \mathbf{B}_0(\mathbf{r})$.

In both cases, $\mathbf{H}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ must satisfy the equations $\operatorname{curl} \mathbf{H} = \mathbf{J}$ and $\operatorname{div} \mathbf{B} = 0$, where $\mathbf{J}(\mathbf{r})$ is the density of the electric currents.

(a) In the case when $\mathbf{H}(\mathbf{r}) = \mathbf{H}_0(\mathbf{r})$, the equation $\operatorname{curl} \mathbf{H} = \operatorname{curl} \mathbf{H}_0 = \mathbf{J} = \mathbf{J}_0$ will be satisfied if $\mathbf{J} = \mathbf{J}_0$. Also, $\mathbf{B} = \mu_r(\mathbf{r})\mathbf{B}_0 \ne \mathbf{B}_0$. From $\operatorname{div} \mathbf{B} = 0$ it follows that $\operatorname{div}(\mu_r \mathbf{B}_0) = (\operatorname{grad} \mu_r) \cdot \mathbf{B}_0 + \mu_r \operatorname{div} \mathbf{B}_0 = (\operatorname{grad} \mu_r) \cdot \mathbf{B}_0 = 0$. This will be satisfied everywhere if and only if $\mathbf{B}_0 \perp \operatorname{grad} \mu_r$. Hence, the permeability can vary only in directions perpendicular to $\mathbf{B}_0$ and $\mathbf{H}_0$. Note that the reference problem and the considered problem should have the same current distribution. (A homogeneous material, for which $\mu_r = \text{const}$, is a special case.)

(b) In the case when $\mathbf{B}(\mathbf{r}) = \mathbf{B}_0(\mathbf{r})$, the equation div $\mathbf{B} = 0$ is automatically satisfied, since div $\mathbf{B}_0 = 0$ in the reference problem, but $\mathbf{H} = \frac{\mathbf{H}_0}{\mu_r} \neq \mathbf{H}_0$. From curl $\mathbf{H} = \mathbf{J}$, it follows that $\text{curl}\left(\frac{\mathbf{H}_0}{\mu_r}\right) = \left(\text{grad}\left(\frac{1}{\mu_r}\right)\right) \times \mathbf{H}_0 + \frac{1}{\mu_r}\text{curl}\,\mathbf{H}_0 = \left(\text{grad}\left(\frac{1}{\mu_r}\right)\right) \times \mathbf{H}_0 + \frac{\mathbf{J}_0}{\mu_r} = \mathbf{J}$. This will be satisfied at each point if and only if $\left(\text{grad}\left(\frac{1}{\mu_r}\right)\right) \times \mathbf{H}_0 + \frac{\mathbf{J}_0}{\mu_r} = \mathbf{J}$. There is no simple mathematical description of the general case in which this condition is fulfilled. For the sake of simplification, let us consider the special case in which $\left(\text{grad}\left(\frac{1}{\mu_r}\right)\right) \times \mathbf{H}_0 = 0$, when the permeability varies only in the direction collinear with $\mathbf{H}_0$ and $\mathbf{B}_0$. The magnetic field exists in the entire space (including the space occupied by the electric currents). If the magnetic material is inhomogeneous, it follows that $\mathbf{J}(\mathbf{r}) = \frac{\mathbf{J}_0(\mathbf{r})}{\mu_r}$ must be satisfied, so that the current distribution is different compared to the distribution in the reference problem, although the current distribution occupies the same space in both cases.

Conclusion

We have outlined the guidelines for teaching hard-to-grasp concepts of analytically solving EM problems that involve inhomogeneous media in electrostatic fields, stationary current fields, and stationary magnetic fields. These problems are the essence of the Fundamentals of Electrical Engineering 1 and 2 coursework, which most of the first-year students in Electrical Engineering have to take. At the introductory level, the coursework focuses on simply recognizing classes of problems that can be solved in closed form. Once recognized, the problems can be solved by applying simple rules, based on comparison with solutions in homogeneous media. In addition, we have presented strict mathematical proofs, based on vector calculus, regarding the types of problems that can be analytically solved. These proofs can be presented in an intermediate level course (e.g. in the third year EM course). Although the presented physical concepts and rigorous mathematical proofs are extremely important for unambiguous comprehension of the fundamentals of EM-field analysis (and as such they are adopted as parts of our regular undergraduate introductory and intermediate level courses in Electrical Engineering), to the best of our knowledge, they have not been addressed previously in such a unified and clear manner in any EM textbooks or educational papers.

Acknowledgement

This work was supported in part by the Serbian Ministry of Education, Science, and Technological Development under grant TR-32005.

Conflict of interest

None declared.

Funding

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

References

1. Haldar MK. Introducing the finite element method in electromagnetics to undergraduates using MATLAB. *Int J Elec Eng Educ* 2006; 43(3): 232–244.
2. Beker B, Bailey DW and Cokkinides GJ. An application-enhanced approach to introductory electromagnetics. *IEEE Trans Educ* 1998; 41(1): 31–36.
3. Collin RE. The role of analysis in the age of computers: view from the analytical side. *IEEE Antennas and Propagation Magazine*, August 1990, pp.27–31.
4. Siakavellas NJ. The importance of analytic solutions for verification: application to the secondary magnetic flux of a conductive disk. *Int J Elec Eng Educ* 2007; 44(4): 342–357.
5. Djordjević AR. *Fundamentals of electrical engineering, parts 1-4*. Belgrade: Academic Mind, 2006 (in Serbian).
6. Božilović G, Olćan D and Djordjević A. *Collection of problems in fundamentals of electrical engineering*, *parts 1-4*. Belgrade: Academic Mind, 2010 (in Serbian).
7. Djordjević AR. *Electromagnetics*. Belgrade: Academic Mind, 2008 (in Serbian).
8. Notaroš BM. *Electromagnetics*. Upper Saddle River, NJ: Prentice Hall, 2011.