TEACHING FUNDAMENTALS OF ELECTRICAL ENGINEERING: NODAL ANALYSIS

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Teaching Fundamentals of Electrical Engineering: Nodal Analysis

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Abstract—**Some aspects of presenting nodal equations to freshmen are outlined. For advanced students, variations of the classical equations are derived, which may be more practical and efficient for hand calculations than the modified nodal analysis.**

Index Terms—**Circuit analysis, Nodal equations**

I. Introduction

$\mathbf{F}$UNDAMENTALS of Electrical Engineering (FEE) 1 and 2 are two first-year courses (at the bachelor level) at the School of Electrical Engineering, University of Belgrade. Each course has 3 hours of lectures and 3 hours of recitations per week, during 15 weeks of a semester (autumn and spring, respectively). For advanced students, supplementary elective courses are offered, with 1 hour of lectures and 1 hour of recitations per week. During the spring semester, there is also a compulsory lab course. The basic course in FEE 1 covers electrostatic fields, steady-current fields, and d.c. circuits. The basic course in FEE 2 covers electromagnetism, a.c. circuits, and transients in simple circuits. The courses serve as a solid introduction to the engineering electromagnetic-field theory and the basic circuit theory. The courses are fully covered by adequate literature [1], [2].

The particular purpose of this paper is to show our approach in the FEE courses to the formulation of the nodal equations and present a simple formulation for networks that contain several branches with only ideal voltage generators in them. In Section II, we present some syllabi of the FEE courses. In Section III, we outline the presentation of the nodal equations. Finally, in Section IV, we describe the modification of the classical nodal equations.

II. FEE Syllabi for d.c. Circuits

We concentrate our attention on the part of FEE 1 which covers d.c. circuits. The syllabi start from the field theory. The current density is introduced first, followed by integral equation for the electric field (which is the basis for KVL) and the continuity equation (the basis for KCL). The constitutive relation is introduced intuitively. The concept of electrical

circuits is derived from the general concept of stationary current fields. Basic d.c. circuit elements are introduced (generators and resistors), along with ideal wires that interconnect them. Voltage-current relations for generators and resistors are deduced, along with power-balance relations. KCL and KVL are derived from the field equations. Ideal voltage and current generators are introduced.

The subsequent syllabi are a solid introduction to the basic circuit theory. They start with tableau equations, from which the reduced form of KVL is deduced. Loop and nodal equations follow. A set of theorems is also presented (linearity, superposition, Thévenen, Norton, reciprocity, maximal power transfer, and bisection). Analysis of circuits with nonlinear resistive elements is covered as well. The course ends with the treatment of networks with capacitors. Advanced students can get acquainted with controlled sources and two-port networks (resistive, with controlled and independent sources). They can also study nonlinear circuits in more depth.

In FEE, we deduce the circuit concept from the electromagnetic-field theory. Hence, we identify a circuit node with an electromagnetic junction, i.e., the node is a point where the current path is divided. Similarly, a branch is a part of the circuit between two nodes, so that it can contain several elements connected in series. Such a choice is convenient for hand calculations, although it is not compatible with the definition of a branch used in the circuit theory.

We visualize a loop as a closed line along circuit branches, and a cut set is defined by considering a closed surface (Gaussian surface) that intersects certain branches. In the FEE courses, we predominantly use nodal cuts.

We have preserved clear distinctions between the potential (V), the voltage ("tension", the difference of potentials, U), and the electromotive force of a voltage generator (emf, E), as in the classical masterpiece [3]. Unfortunately, in some modern textbooks [4], these three quantities are used without clear distinction.

III. Classical Nodal Equations

The first step in the FEE tuition is to figure out the independent "voltage-type" variables [1]. We consider circuits with connected graphs, so that the number of these variables equals the number of nodes less one ($n_\mathrm{f} = n_\mathrm{n} - 1$). Adopting one node to be the reference (grounded) node, we identify the potentials of the remaining (non-grounded, hot) nodes to form a set of independent variables.

Consider a resistive circuit that consists of resistors and independent sources, but without zero-resistance branches (i.e., no branch is an ideal voltage source). We write KCL for the hot nodes, express the branch currents in terms of the node potentials, and rearrange the resulting equations to yield the basic form of the nodal equations:

$$G_{11}V_1 + G_{12}V_2 + ... + G_{1n_f}V_{n_f} = I_{n1}$$
$$G_{21}V_1 + G_{22}V_2 + ... + G_{2n_f}V_{n_f} = I_{n2}$$
$$\vdots \qquad\qquad\qquad\qquad , \qquad (1)$$
$$G_{n_f 1}V_1 + G_{n_f 2}V_2 + ... + G_{n_f n_f}V_{n_f} = I_{nn_f}$$

where

- $G_{ii}, i = 1,...,n_f$, is the sum of conductances of all branches that are connected to node i,
- $G_{ij}, i, j = 1,...,n_f, i \neq j$, is the sum of conductances of all branches that directly interconnect nodes i and j, multiplied by -1, and
- $I_{ni}, i = 1,...,n_f$, is the algebraic sum of currents of all current and equivalent current generators of branches connected to node i:
  - an equivalent current generator is obtained by transformation of a real voltage generator into a real current generator;
  - when the reference direction of a current is towards the node, the current is taken as is, while it is multiplied by -1 otherwise.

In FEE, the students are trained to write the nodal equations for a given circuit directly, by inspecting the circuit and using the above template.

Next, we consider the case when in one branch there is only an ideal voltage generator. The nodal equations (1) cannot be formulated because there exists a zero-resistance (infinite-admittance) branch. We bypass the problem by taking one node to which this generator is connected to be the reference node. Now, the potential of the other node is known as it is determined by the generator emf. We generalize this approach to the case when there are several branches that contain only ideal voltage generators, but all these generators are connected to one (grounded) node.

For advanced students, we go on. We introduce the modified nodal analysis (MNA). We stress the usefulness of the method in modern circuit simulation tools (e.g., the Spice family) [5]. However, we also point out the potential deficiencies [6], [7]: for each ideal voltage generator, we must artificially add one unknown and thus increase the order of the system of linear equations that is to be solved. This augmenting is inconvenient for hand calculations, and may not be the optimal choice for computer-aided solutions as well.

IV. NEW FORMULATION OF NODAL EQUATIONS

For circuits that contain several branches with only ideal voltage generators, a more intuitive approach than MNA, at least for the freshmen, may be to do simple and straightforward manipulations with the nodal equations and thus solve the circuit, as demonstrated in this section. This approach yields the same equations as the ultimate set obtained when applying the supernode concept [8], [9]. However, instead of deriving the equations from scratch each time a circuit is solved, we define a template for directly writing down the appropriate set of equations.

We consider two examples, shown in Fig. 1. Node 0 is assumed to be the reference. For both circuits, we let $R_1, R_2 \rightarrow 0$, so that we end up with ideal voltage generators in two branches (with element indices 1 and 2). In the first example, these two generators are interconnected in a chain. In the second example, the two generators are not connected to common nodes.
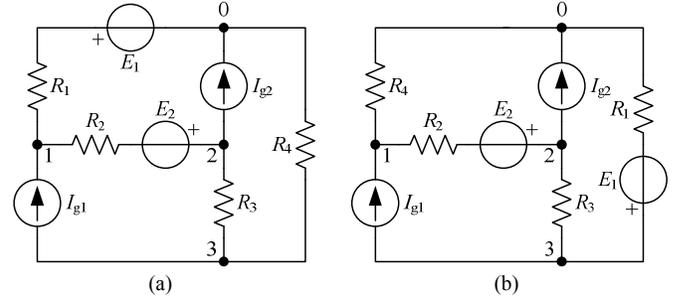
Figure 1. Examples of circuits that contain branches with only ideal voltage generators when $R_1, R_2 \rightarrow 0$ [2].

A. Chained Generators

Nodal equations for the first example (Fig. 1a), when $R_1, R_2 > 0$, read

$$\left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_1 - \frac{1}{R_2}V_2 = \frac{E_1}{R_1} - \frac{E_2}{R_2} + I_{g1}, \qquad (2)$$

$$-\frac{1}{R_2}V_1 + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)V_2 - \frac{1}{R_3}V_3 = \frac{E_2}{R_2} - I_{g2}, \qquad (3)$$

$$-\frac{1}{R_3}V_2 + \left(\frac{1}{R_3} + \frac{1}{R_4}\right)V_3 = -I_{g1}, \qquad (4)$$

We multiply equation (3) by R_2 and then take $R_2 \rightarrow 0$. The result is

$$-V_1 + V_2 = E_2. \qquad (5)$$

From (2) alone, we cannot conclude what happens when $R_1, R_2 \rightarrow 0$. However, if we eliminate the potential V_2 from (2) by using (5), the terms containing $\frac{1}{R_2}$ cancel out. After multiplication by R_1, in the limit when $R_1 \rightarrow 0$, we obtain

$$V_1 = E_1. \qquad (6)$$

Equations (4)-(6) form a system of linear equations. Obviously, $V_1 = E_2$ and $V_2 = E_1 + E_2$, so that V_3 can now be evaluated from (4).

This procedure can be generalized to an arbitrary case when ideal voltage generators are chained. We take the beginning of the chain to be the reference point. We do not write classical nodal equations for the nodes at which the ideal voltage generators are connected. However, the potentials of these

nodes are obtained by algebraically summing the corresponding emfs. For the remaining hot nodes, we write classical nodal equations, in which some potentials are already known. Hence, we can obtain the remaining potentials.

B. Arbitrary Case

For the circuit shown in Fig. 1b, assuming $R_1, R_2 > 0$, the nodal equations read

$$\left(\frac{1}{R_4} + \frac{1}{R_2}\right)V_1 - \frac{1}{R_2}V_2 = -\frac{E_2}{R_2} + I_{g1}, \tag{7}$$

$$-\frac{1}{R_2}V_1 + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)V_2 - \frac{1}{R_3}V_3 = \frac{E_2}{R_2} - I_{g2}, \tag{8}$$

$$-\frac{1}{R_3}V_2 + \left(\frac{1}{R_1} + \frac{1}{R_3}\right)V_3 = \frac{E_1}{R_1} - I_{g1}. \tag{9}$$

When $R_1 \to 0$, from (9) we obtain

$$V_3 = E_1. \tag{10}$$

When $R_2 \to 0$, (7) yields

$$V_1 - V_2 = -E_2. \tag{11}$$

However, the same result also follows from (8). Hence, we do not obtain a sufficient number of independent equations ($n_n - 1 = 3$) to solve the problem. The remedy is to add (7) and (8), which yields

$$\frac{1}{R_4}V_1 + \frac{1}{R_3}V_2 - \frac{1}{R_3}V_3 = I_{g1} - I_{g2}. \tag{12}$$

The system of equations consists now of (10), (11), and (12), which is simple to be solved.

We generalize here this procedure, as follows. We consider a circuit, like the example in Fig. 2, where some branches contain only ideal voltage generators. We refer to these branches as zero-resistance branches (ZRBs). Such branches may be clustered. For the circuit in Fig. 2, there are two clusters.
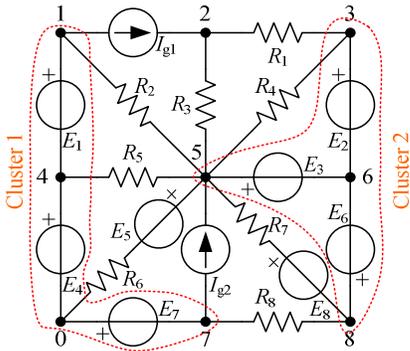
Figure 2. Example of a circuit with two ZRB clusters.

The ZRBs in a cluster must not form closed loops. If they do, such zero-resistance loops would make the solution impossible (if the algebraic sum of emfs along a loop is not zero) or undetermined (if the algebraic sum is zero). Hence, the ZRBs in a cluster can only be arranged in the form of a sub-tree. Note that standard circuit simulators do perform topological tests while searching for zero-resistance loops.

Such tests already have sufficient information to find the corresponding ZRB sub-trees at practically no additional computational cost.

Note that the total number of nodes belonging to a ZRB cluster with z generators ($z \geq 1$) equals $z + 1$.

Skipping the derivation, which is based on the same guidelines as for the circuit shown in Fig. 1b, we immediately give rules for writing down the system of nodal equations.

For each hot node that does not belong to a ZRB cluster, we write an equation in the classical form (1). For the circuit in Fig. 2, there is only one such node (node 2).

Instead of the classical equations for the first z nodes of a cluster, we write an equation of the form

$$V_i - V_j = E_{ji}, \tag{13}$$

where E_{ji} is the emf of the ideal voltage generator in the branch that connects nodes i and j. If one of these two nodes is grounded, its potential does not appear in (13). We write equations of the form (13) for all ZRBs in the cluster.

For a cluster that does not encompass the grounded node, instead of the classical equation for the last node of the cluster (whose index is i_c), we write an equation of the form

$$G_{i_c 1}V_1 + G_{i_c 2}V_2 + ... + G_{i_c n_f}V_{n_f} = I_{ni_c}, \tag{14}$$

where I_{ni_c} is the algebraic sum of currents of all current generators and equivalent current generators of branches connected to the cluster, excluding the ZRBs and other branches that interconnect two nodes of the cluster. If node j belongs to the cluster, $G_{i_c j}$ is the sum of conductances of all branches that are connected to that node, excluding the ZRBs and other branches that directly interconnect to other nodes of the cluster. If node j does not belong to the cluster, $G_{i_c j}$ is the sum of conductances of all branches that are connected between that node and the cluster, multiplied by -1.

For a cluster whose one node is grounded, there is no equation of the form (14).

Following these rules, the system of nodal equations for the circuit shown in Fig. 2 is obtained as follows. The first cluster encompasses nodes 1, 4, 0, and 7. The second cluster encompasses nodes 3, 5, 6, and 8. For node 2, we have a classical equation. Equations at locations reserved for nodes 1, 4, and 7, as well as for 3, 5, and 6, have the form (13). Equation for node 8 has the form (14). Hence, we have the following system of simultaneous linear equations:

$$V_1 - V_4 = E_1, \tag{15.1}$$

$$\left(\frac{1}{R_1} + \frac{1}{R_3}\right)V_2 - \frac{1}{R_1}V_3 - \frac{1}{R_3}V_5 = I_{g1}, \tag{15.2}$$

$$V_3 - V_6 = E_2, \tag{15.3}$$

$$V_4 = E_4, \tag{15.4}$$

$$V_5 - V_6 = E_3, \tag{15.5}$$

$$V_6 - V_8 = -E_6, \tag{15.6}$$

$$V_7 = -E_7, \tag{15.7}$$

$$-\frac{1}{R_2}V_1 - \left(\frac{1}{R_1}+\frac{1}{R_3}\right)V_2 + \frac{1}{R_1}V_3 - \frac{1}{R_5}V_4$$

$$+\left(\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_5}+\frac{1}{R_6}\right)V_5 - \frac{1}{R_8}V_7 + \frac{1}{R_8}V_8 = \frac{E_5}{R_6} + I_{g2} \,. \quad (15.8)$$

As an explanation for (15.8), node 1 is connected to node 5 of cluster 2 by the resistor R_2. A similar situation is for nodes 4 and 7. Node 2 is connected to nodes 3 and 5 of the cluster by R_1 and R_3. Node 3 is in cluster 2 and it is connected only to node 2 outside the cluster. A similar situation is for node 8. Node 5 is connected to nodes 1, 2, 4, 7, and 0 outside cluster 2. The resistor R_4 and the branch E_8-R_7 interconnect nodes that belong to cluster 2. Hence, R_4, R_7, and E_8, do not appear in the above equations. By inspecting the scheme in Fig. 2, we can see that these elements, indeed, cannot affect the node potentials, although they do affect the currents of E_2, E_3, and E_6.

Note that we have ended up with a total of $n_f = n_n - 1 = 8$ equations.

For comparison, to write the classical MNA equations for the circuit of Fig. 2, we have to introduce currents of the six ideal voltage generators, as shown in Fig. 3. We end up with a grand total of 14 simultaneous linear equations:

$$\frac{1}{R_2}V_1 - \frac{1}{R_2}V_5 - I_{E1} = -I_{g1}, \quad (16.1)$$

$$\left(\frac{1}{R_1}+\frac{1}{R_3}\right)V_2 - \frac{1}{R_1}V_3 - \frac{1}{R_3}V_5 = I_{g1}, \quad (16.2)$$

$$-\frac{1}{R_1}V_2 + \left(\frac{1}{R_1}+\frac{1}{R_4}\right)V_3 - \frac{1}{R_4}V_5 - I_{E2} = 0, \quad (16.3)$$

$$\frac{1}{R_5}V_4 - \frac{1}{R_5}V_5 + I_{E1} - I_{E4} = 0, \quad (16.4)$$

$$-\frac{1}{R_2}V_1 - \frac{1}{R_3}V_2 - \frac{1}{R_4}V_3 - \frac{1}{R_5}V_4$$

$$+\left(\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4}+\frac{1}{R_5}+\frac{1}{R_6}+\frac{1}{R_7}\right)V_5$$

$$-\frac{1}{R_7}V_8 - I_{E3} = \frac{E_5}{R_6} + \frac{E_8}{R_7} + I_{g2}, \quad (16.5)$$

$$I_{E2} + I_{E3} + I_{E6} = 0, \quad (16.6)$$

$$\frac{1}{R_8}V_7 - \frac{1}{R_8}V_8 + I_{E7} = -I_{g2}, \quad (16.7)$$

$$-\frac{1}{R_7}V_5 - \frac{1}{R_8}V_7 + \left(\frac{1}{R_7}+\frac{1}{R_8}\right)V_8 - I_{E6} = -\frac{E_8}{R_7}, \quad (16.8)$$

$$V_1 - V_4 = E_1, \quad (16.9)$$

$$V_3 - V_6 = E_2, \quad (16.10)$$

$$V_5 - V_6 = E_3, \quad (16.11)$$

$$V_4 = E_4, \quad (16.12)$$

$$V_8 - V_6 = E_6, \quad (16.13)$$

$$V_7 = -E_7 \,. \quad (16.14)$$

By comparing systems (15) and (16), we can see that the proposed method leads to a substantially smaller and simpler system of linear equations than the classical MNA, which is the essential advantage for hand calculations. The only benefit of MNA is that the currents of the ideal voltage generators are readily available, which is paid by a price of a larger system. In MNA, the currents of all other branches are obtained by *a posteriori* calculations.

Similarly, in the proposed method, all branch currents are obtained by *a posteriori* calculations. In particular, the currents of the ideal voltage generators are evaluated using KCL.
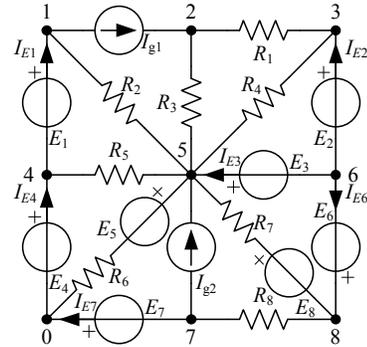
Figure 3. Notation for classical MNA equations.

V. Conclusion

We have presented the basic steps in teaching the nodal equations for the freshmen, within the Fundamentals of Electrical Engineering.

In addition, we have developed an alternative to the classical MNA approach for circuits that contain zero-resistance branches. The proposed method is more intuitive and convenient for hand calculations because it is based on a smaller and simpler system of linear equations than MNA. The technique is introduced for d.c. circuits. However, it is directly applicable to a.c. circuits and transient analysis.

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